Things We Do That Can't Be Done - 2
Science/Math Illegalities in Engineering
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Engineering ranks efficiency and practicality over scientific or mathematical perfection in our calculations. Especially when the idealized mathematical perfection does not exactly match how Nature behaves. However, our convention with terms and equations often can lead others astray when the situation is not the same as the one that justifies shortcuts. Those more grounded in mathematical perfection might call us downright sloppy! This article, Part II in the series, discusses the substitution of frequency for probability and omission of terms in propagation of probability in OR situations.

Probability:

Probability is the proportion (ratio) of the number of occurrences of an event in a given number of trials that could generate the event. It could either be a theoretically expected proportion. For instance, the probability of getting a "4" when rolling a standard 6-sided die is 1/6 = 0.16666, which would be stated as p(a "4" on one dice roll)=0.16666. This is a theoretical value based on each side of the die having the same chance of showing on top. If you roll the die 6,000,000 times you expect to get the event of a "4" to happen 1,000,000 times. Then p=1,000,000/6,000,000=0.16666… This is the ideal "true" value, true only if the roll is randomized and the equal likelihood expectation of any number on top is true. In reality, slight variation on the die dimensions or where the "4" is positioned in initiating the roll could make one number more favorable. The theoretical probability is for an imagined reality. It would be an estimate for reality.

However, even if all the idealizations were true, you should not expect to get exactly 1/6th of the outcomes to be a "4". Consider 10 rolls. 1/6th of 10 is 1.6666. You cannot get 2/3^{rds} of an event. The outcome is either an event or not an event. Consider 600 rolls. You expect 100 "4" events but would not be surprised if the random aspect of the trials ended with 98 "4"s or 103 "4"s. Nature will not give you exactly the expected number.

There might not be a conceptual basis to theoretically determine the probability. In this case, you could observe past data (trials). For instance, roll the die 100 times and count the number of "4" events. Maybe it is 18. Maybe 13. In any case, the estimate of the probability from past data would be 0.18 or 0.13. This empirical value is also not the true probability value. It is an estimate from a limited number of trials.

Although useful, neither method to determine probability returns an exactly true value.

Note a trial (such as a die roll) cannot be cut in half. A trial is a completed batch of procedures (pick up the die, shake it to randomize its orientation, toss it to roll on the table, when it comes to rest observe the number). You cannot have an event outcome from half a roll. This scaling impossibility will guide you to alternate ways to analyze probability.

Frequency

Often we use the term probability to quantify the number of events that might (or did) happen in a time or space interval. But this is a frequency, or a rate, not a probability. For instance, if the expectation is two events in one year, the frequency is 2 events per year. But this is not the probability. For one reason, probability must be between 0 and 1, inclusive. One might choose to reduce the time interval to a quarter-of-a-year, and say that the expectation is 0.5 events per quarter. Then claim the probability is 0.5 which is a legitimate value. But don't. Based on a one-week time interval the rate is 2/52=0.03846… If probability is frequency, one choice makes the probability 2, another 0.5, and another 0.03846. If the year is considered the trial, you can halve it. But with probability you cannot have half of a trial. Again, events per year is not a probability. One clue that frequency is not probability is that you can divide the trial period to half a trial period.

Ideally, the Poisson distribution converts the expected frequency of events (number of events per year, or per length, or per area) to the probability of a number of events within that interval. The minimum number is zero, but there is no upper bound on the possible number of events. The idealization in the Poisson model is that any event is wholly independent of any other event.

Use the expected or average rate or frequency (here 2/year) as lambda, λ, in a Poisson model. $P(x) = \frac{\lambda^n e^{-\lambda}}{n!}$. n is the number of possible events in the time interval, and P(n) is the probability of that number of events. For instance, if the expected rate is $\lambda = 2\ events/yr$, then the probability of 0,1,2,3,… events is in Table 1.

Table 1 – Probability of number of events per year

n Events/yr	P(n events/yr) (rounded)	P(n events/yr) (rounded)
	$\lambda = 2\ events/yr$	$\lambda = 0.01\ events/yr$
0	$P(0) = \frac{2^0 e^{-2}}{0!} = 0.135$	0.99005
1	$P(1) = \frac{2^1 e^{-2}}{1!} = 0.271$	0.00990
2	$P(2) = \frac{2^2 e^{-2}}{2!} = 0.271$	4.95E-05
3	$P(3) = \frac{2^3 e^{-2}}{3!} = 0.180$	1.65E-07

4	$P(4) = \frac{2^4 e^{-2}}{4!} = 0.090$	4.12E-10
5	$P(5) = \frac{2^5 e^{-2}}{5!} = 0.036$	8.25E-13

There is a probability of zero events in a year; also 2, 3, 4, 5 or more events per year.

If the expectation is 2 events per year, this does not mean that in any one year you will experience two events. You might not have any. The probability of no events in a year is 0.135. You might have 4. The probability of 4 events in one year is 0.090. The probability of one event in a year is 0.271 which is not equal to the average frequency of 2 events/yr.

If, however the expected frequency is very small, for instance if $\lambda = 1\ event\ in\ 100\ years = 0.01/year$, then the probabilities of 2 or more events in a year are vanishingly small, and the probability of 1 event in a year is $P(1) = \frac{0.01^1 e^{-0.01}}{1!} = 0.099 \cong 0.01 = \lambda$. See Table 1. If frequency is small, frequency can approximate the probability.

Since the frequencies and probabilities estimated from past and other processes, under different management conditions, are just estimates, and since the assumption that events are truly independent (no common cause), the error on equating probability to frequency is justifiable (when frequency is small). This convenience introduces an error, but the error is negligible relative to the uncertainty of the givens and the idealization in the model, so there is no need to use perfect analysis.

If the expectation of the number of events scales with the duration of time or magnitude of the area (If you double the time, or double the area, you expect twice as many events), then use the Poisson distribution.

Multiple Trials

The expectation is that in 4 coin flips (4 independent trials), 2 will be Heads because the probability of getting a Head in any one flip is 0.5. But there might be no Heads in the 4 trials, or 1 or 2 or 3 or 4 Head outcomes in the set of flips.

Here the expected number of events scales with the number of independent trials. However, the number of possible events cannot exceed the number of trials. This limit contrasts with the unlimited number of possible events that characterizes a Poisson distribution. The multiple independent trials and limit on the possible number of events would be a clue to use the binomial distribution.

The binomial distribution is:

$$P(n|N) = \frac{N!}{(N-n)!\,n!} p^n q^{N-n}$$

Here $P(n|N)$ means the probability of n number of events in N trials. p is the probability of an event in one trial, and $q = 1 - p$ is the probability of not an event. The model idealization is that the events are wholly independent.

Ideally the binomial distribution provides the probability of n number of events in N number of trials. For the coin flips the probability of an event (a Head) in a trial (a flip) is 0.5. And for any of 5 possible outcomes, Table 2 indicates the ideal probability.

Table 2 – The probability of n events in N trials

n Events in N=4 Trials	P(n|N)	P(n|N)
	p=.5	p=0.001
0	0.0625	0.996006
1	0.250	0.003988
2	0.375	5.988E-06
3	0.250	3.996E-09
4	0.0625	1E-12

Again, in the limit of low probability, such as a 0.001 chance of having an event in one trial, the probability of one event in 4 trials is about $0.003988 \cong 0.004 = 4 * 0.001 = 4 * p$. Again, with very small event probabilities the probability of an event scales with the number of trials. But this is not a mathematical truth.

The AND conjunction

Logical AND conjunctions and reliability gates multiply the individual event probabilities. Ideally, if the probabilities of events A and B are independent of each other.

$$P(A\ AND\ B) = P(A) \cdot P(B)$$

For example: The probability of rolling a “3” on one die (p=1/6) AND flipping Head on a coin (p=1/2) is 1/6*1/2=1/12=0.08333. In context of a process reliability analysis if frequency is substituted for probability (because it is a small value), and there are two full-sized pumps in parallel, the process stops if both fail. If the probability of one failing is 0.02 per year, if there are no common cause relations, and if it takes a month to repair/replace a failed pump, then the probability of both failing at the same time in a year is 0.02*0.02/12=0.00003333.

The OR conjunction

Logical OR conjunctions and reliability gates are a bit more complicated. One way to obtain the OR rule is to use the equivalent NOT condition. $P(NOT\ A) = 1 - P(A)$. Then $P(A\ OR\ B) = 1 - P(NOT\ (A\ OR\ B)) = 1 - P(NOT\ A\ AND\ NOT\ B) = 1 - \left(1 - P(A)\right)\left(1 - P(B)\right) = P(A) + P(B) - P(A)P(B)$. So

$$P(A\ OR\ B) = P(A) + P(B) - P(A)P(B)$$

For example: The probability of rolling a "3" on one die (p=1/6) OR flipping Head on a coin (p=1/2) is 1/6+1/2-1/6*1/2=1/12=0.58333.

If the individual probabilities are small, then the product $P(A)P(B)$ will be vanishingly small and its impact can be ignored. Further, considering that the individual $P(A)$ and $P(B)$ values are not known with certainty, and the complete independence of events A and B may not be true, the uncertainty on the givens and model idealization may be much larger than the contribution of discarding $P(A)P(B)$. So conventionally, in reliability and safety analysis (with small frequency values substituted for probability), the OR conjunction rule is shortened to:

$$P(A\ OR\ B) \cong P(A) + P(B)$$

Consider another example: If the probability of rain is 60% this afternoon and 70% tonight, what is the probability of rain today. Using the truncated OR rule, $P(rain\ today) = P(rain\ afternoon\ OR\ rain\ tonight) \cong P(rain\ afternoon) + P(rain\ tonight) = 0.6 + 0.7 = 1.3 = 130\%$. But this cannot be, because the probability exceeds 100%. By contrast, $P(rain\ afternoon\ OR\ rain\ tonight) = P(rain\ afternoon) + P(rain\ tonight) - P(rain\ afternoon\ AND\ night) = 0.6 + .07 - (0.6)(0.7) = 0.88 = 88\%$.

Takeaway

Keep aware of the difference between probability and frequency, and the independence idealization in the calculations. You cannot have half a trial. In spite of what "they" call it, if the statement of "probability" permits you to cut the trial period or area in half, you have a frequency not a probability. Only use the truncated OR model when frequency or probabilities are low.

For greater detail, consider my book Applied Engineering Statistics 2nd Edition (Rhinehart and Bethea) Taylor & Francis CRC press, ISBN: 9781032119489.

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